Projective Geometric Algebra

projectivegeometricalgebra.org

Basis Elements

Type	Values	Grade / Antigrade
Scalar	1	0 / 4
Vectors	$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4 = \mathbf{e}_n$	1 / 3
Bivectors	$\mathbf{e}_{41} = \mathbf{e}_4 \wedge \mathbf{e}_1$ $\mathbf{e}_{42} = \mathbf{e}_4 \wedge \mathbf{e}_2$ $\mathbf{e}_{43} = \mathbf{e}_4 \wedge \mathbf{e}_3$ $\mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{31} = \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$	2 / 2
Trivectors / Antivectors	$\mathbf{e}_{423} = \mathbf{e}_4 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{431} = \mathbf{e}_4 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{412} = \mathbf{e}_4 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2$ $\mathbf{e}_{321} = \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1$	3 / 1
Antiscalar	$\mathbb{1} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$	4 / 0

Metric

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathbb{G}\mathbf{u} = \overline{\mathbf{G}\underline{\mathbf{u}}} = \underline{\mathbf{G}\overline{\mathbf{u}}}$ $\qquad \mathbf{G}(\mathbf{a} \wedge \mathbf{b}) = \mathbf{G}\mathbf{a} \wedge \mathbf{G}\mathbf{b}$

$\mathbf{G}\mathbb{G} = \det(\mathbf{g})\,\mathbf{I}$ $\qquad \mathbb{G}(\mathbf{a} \vee \mathbf{b}) = \mathbb{G}\mathbf{a} \vee \mathbb{G}\mathbf{b}$

$$\mathbf{G} = \operatorname{diag}\left(1;\ \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix};\ \begin{bmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1 \end{bmatrix};\ \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{bmatrix};\ 0\right)$$

Unary Operations

Operation	Description	Identities
$\overline{\mathbf{u}}$	Right complement of $\mathbf{u}$	$\mathbf{u} \wedge \overline{\mathbf{u}} = \mathbb{1}$ $\quad \mathbf{u} \vee \overline{\mathbf{u}} = 1$
$\underline{\mathbf{u}}$	Left complement of $\mathbf{u}$	$\underline{\mathbf{u}} \wedge \mathbf{u} = \mathbb{1}$ $\quad \underline{\mathbf{u}} \vee \mathbf{u} = 1$ $\underline{\mathbf{u}} = (-1)^{\operatorname{gr}(\mathbf{u})\operatorname{ag}(\mathbf{u})}\,\overline{\mathbf{u}}$
$\mathbf{u}_\bullet = \mathbf{G}\mathbf{u}$	Bulk of $\mathbf{u}$	$\mathbf{u} = \mathbf{u}_\bullet + \mathbf{u}_\circ$ $\mathbf{u}_\bullet = (\mathbf{e}_n \wedge \mathbf{u}) \vee \overline{\mathbf{e}}_n$
$\mathbf{u}_\circ = \mathbb{G}\mathbf{u}$	Weight of $\mathbf{u}$	$\mathbf{u}_\circ = \mathbf{e}_n \wedge (\mathbf{u} \vee \overline{\mathbf{e}}_n)$
$\mathbf{u}^\star = \overline{\mathbf{G}\mathbf{u}}$	Right bulk dual of $\mathbf{u}$	$\mathbf{u}^\star = \overline{\mathbf{u}_\bullet}$ $\quad \mathbf{u}^\star = \tilde{\mathbf{u}} ⟑ \mathbb{1}$
$\mathbf{u}^{☆} = \overline{\mathbb{G}\mathbf{u}}$	Right weight dual of $\mathbf{u}$	$\mathbf{u}^{☆} = \overline{\mathbf{u}_\circ}$ $\quad \mathbf{u}^{☆} = \underset{\sim}{\mathbf{u}} ⟇ 1$
$\mathbf{u}_\star = \underline{\mathbf{G}\mathbf{u}}$	Left bulk dual of $\mathbf{u}$	$\mathbf{u}_\star = \underline{\mathbf{u}_\bullet}$ $\quad \mathbf{u}_\star = \mathbb{1} ⟑ \tilde{\mathbf{u}}$
$\mathbf{u}_{☆} = \underline{\mathbb{G}\mathbf{u}}$	Left weight dual of $\mathbf{u}$	$\mathbf{u}_{☆} = \underline{\mathbf{u}_\circ}$ $\quad \mathbf{u}_{☆} = 1 ⟇ \underset{\sim}{\mathbf{u}}$
$\tilde{\mathbf{u}}$	Reverse of $\mathbf{u}$	$\tilde{\mathbf{u}} = (-1)^{\operatorname{gr}(\mathbf{u})(\operatorname{gr}(\mathbf{u})-1)/2}\,\mathbf{u}$
$\underset{\sim}{\mathbf{u}}$	Antireverse of $\mathbf{u}$	$\underset{\sim}{\mathbf{u}} = (-1)^{\operatorname{ag}(\mathbf{u})(\operatorname{ag}(\mathbf{u})-1)/2}\,\mathbf{u}$ $\underset{\sim}{\mathbf{u}} = (-1)^{\operatorname{gr}(\mathbf{u})\operatorname{ag}(\mathbf{u}) + \frac{n(n-1)}{2}}\,\tilde{\mathbf{u}}$

Binary Operations

Operation	Description	Identities
$\mathbf{a} \wedge \mathbf{b}$	Exterior product Wedge product a "wedge" b	$\overline{\mathbf{a} \wedge \mathbf{b}} = \overline{\mathbf{a}} \vee \overline{\mathbf{b}}$ $\overline{\mathbf{a} \vee \mathbf{b}} = \overline{\mathbf{a}} \wedge \overline{\mathbf{b}}$
$\mathbf{a} \vee \mathbf{b}$	Exterior antiproduct Antiwedge product a "antiwedge" b	$\mathbf{a} \wedge \mathbf{b} = (-1)^{\operatorname{gr}(\mathbf{a})\operatorname{gr}(\mathbf{b})}\,\mathbf{b} \wedge \mathbf{a}$ $\mathbf{a} \vee \mathbf{b} = (-1)^{\operatorname{ag}(\mathbf{a})\operatorname{ag}(\mathbf{b})}\,\mathbf{b} \vee \mathbf{a}$
$\mathbf{a} \bullet \mathbf{b}$	Inner product Dot product a "dot" b	$\mathbf{a} \bullet \mathbf{b} = (\mathbf{a}^{\mathrm{T}}\mathbf{G}\mathbf{b})\,\mathbf{1}$ $\mathbf{a} \circ \mathbf{b} = (\mathbf{a}^{\mathrm{T}}\mathbb{G}\mathbf{b})\,\mathbb{1}$
$\mathbf{a} \circ \mathbf{b}$	Inner antiproduct Antidot product a "antidot" b	$\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$ $\mathbf{a} \circ \mathbf{b} = \mathbf{b} \circ \mathbf{a}$
$\mathbf{a} ⟑ \mathbf{b}$	Geometric product a "wedge-dot" b Identity is scalar $\mathbf{1}$	$\overline{\mathbf{a} ⟑ \mathbf{b}} = \overline{\mathbf{a}} ⟇ \overline{\mathbf{b}}$ $\overline{\mathbf{a} ⟇ \mathbf{b}} = \overline{\mathbf{a}} ⟑ \overline{\mathbf{b}}$
$\mathbf{a} ⟇ \mathbf{b}$	Geometric antiproduct a "antiwedge-dot" b Identity is antiscalar $\mathbb{1}$	
$\mathbf{a} \vee \mathbf{b}^\star$	Bulk contraction	$\mathbf{a} \vee (\mathbf{b} \wedge \mathbf{c})^\star = \mathbf{a} \vee \mathbf{b}^\star \vee \mathbf{c}^\star$ $\mathbf{a} \wedge (\mathbf{b} \vee \mathbf{c})^{☆} = \mathbf{a} \wedge \mathbf{b}^{☆} \wedge \mathbf{c}^{☆}$
$\mathbf{a} \vee \mathbf{b}^{☆}$	Weight contraction	
$\mathbf{a} \wedge \mathbf{b}^\star$	Bulk expansion	$\mathbf{a} \vee \mathbf{b}^\star = \mathbf{a} \bullet \mathbf{b}$, when $\operatorname{gr}(\mathbf{a}) = \operatorname{gr}(\mathbf{b})$
$\mathbf{a} \wedge \mathbf{b}^{☆}$	Weight expansion	$\mathbf{a} \wedge \mathbf{b}^{☆} = \mathbf{a} \circ \mathbf{b}$, when $\operatorname{ag}(\mathbf{a}) = \operatorname{ag}(\mathbf{b})$

$(\mathbf{a} \vee \mathbf{b}^\star) \bullet \mathbf{c} = \mathbf{a} \bullet (\mathbf{b} \wedge \mathbf{c})$, when $\operatorname{gr}(\mathbf{a}) = \operatorname{gr}(\mathbf{b}) + \operatorname{gr}(\mathbf{c})$

$(\mathbf{a} \wedge \mathbf{b}^{☆}) \circ \mathbf{c} = \mathbf{a} \circ (\mathbf{b} \vee \mathbf{c})$, when $\operatorname{ag}(\mathbf{a}) = \operatorname{ag}(\mathbf{b}) + \operatorname{ag}(\mathbf{c})$

$$\left[(\mathbf{a} \wedge \mathbf{b}) \vee \mathbf{v}^\star\right]_\bullet = \left[(\mathbf{a} \vee \mathbf{v}^\star) \wedge \mathbf{b} + (-1)^{\operatorname{gr}(\mathbf{a})}\,\mathbf{a} \wedge (\mathbf{b} \vee \mathbf{v}^\star)\right]_\bullet$$

$$\left[(\mathbf{a} \vee \mathbf{b}) \wedge \mathbf{v}^{☆}\right]_\circ = \left[(\mathbf{a} \wedge \mathbf{v}^{☆}) \vee \mathbf{b} + (-1)^{\operatorname{ag}(\mathbf{a})}\,\mathbf{a} \vee (\mathbf{b} \wedge \mathbf{v}^{☆})\right]_\circ$$

Norms

Definition	Description
$\lVert\mathbf{u}\rVert_\bullet = \sqrt{\mathbf{u} \bullet \mathbf{u}}$	Bulk norm of $\mathbf{u}$
$\lVert\mathbf{u}\rVert_\circ = \sqrt{\mathbf{u} \circ \mathbf{u}}$	Weight norm of $\mathbf{u}$
$\lVert\mathbf{u}\rVert = \lVert\mathbf{u}\rVert_\bullet + \lVert\mathbf{u}\rVert_\circ = \sqrt{\mathbf{u} \bullet \mathbf{u}} + \sqrt{\mathbf{u} \circ \mathbf{u}}$	Geometric norm of $\mathbf{u}$

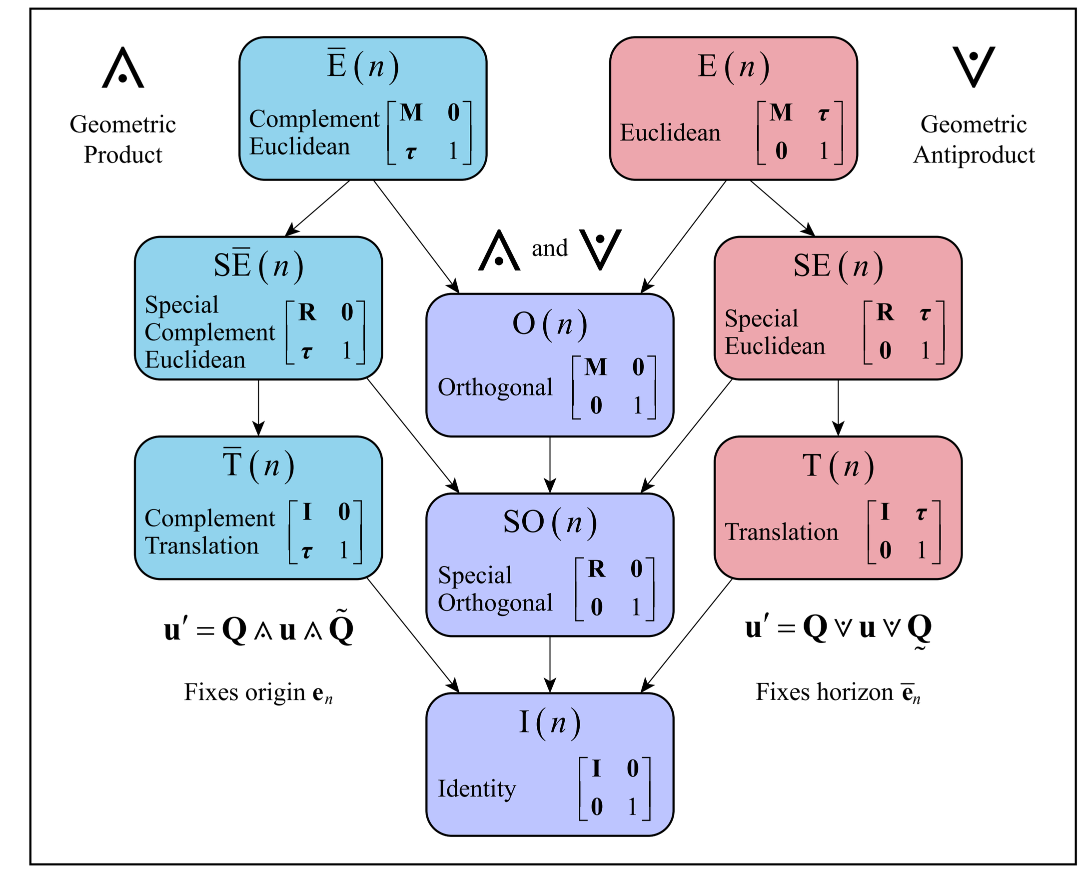
Definition: $\widehat{\lVert\mathbf{u}\rVert} = \dfrac{\lVert\mathbf{u}\rVert_\bullet}{\lVert\mathbf{u}\rVert_\circ} = \dfrac{\sqrt{\mathbf{u} \bullet \mathbf{u}}}{\sqrt{\mathbf{u} \circ \mathbf{u}}}$ — Projected geometric norm of $\mathbf{u}$

Type	Projected Geometric Norm	Interpretation
Point $\mathbf{p}$	$\widehat{\lVert\mathbf{p}\rVert} = \dfrac{\sqrt{p_x^2 + p_y^2 + p_z^2}}{\lvert p_w \rvert}$	Distance from origin to point $\mathbf{p}$. Half distance that origin is moved by flector $\mathbf{p}$.
Line $\boldsymbol{l}$	$\widehat{\lVert\boldsymbol{l}\rVert} = \dfrac{\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}}{\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}}$	Perpendicular distance from origin to line $\boldsymbol{l}$. Half distance that origin is moved by motor $\boldsymbol{l}$.
Plane $\mathbf{g}$	$\widehat{\lVert\mathbf{g}\rVert} = \dfrac{\lvert g_w \rvert}{\sqrt{g_x^2 + g_y^2 + g_z^2}}$	Perpendicular distance from origin to plane $\mathbf{g}$. Half distance that origin is moved by flector $\mathbf{g}$.
Motor $\mathbf{Q}$	$\widehat{\lVert\mathbf{Q}\rVert} = \sqrt{\dfrac{Q_{mx}^2 + Q_{my}^2 + Q_{mz}^2 + Q_{mw}^2}{Q_{vx}^2 + Q_{vy}^2 + Q_{vz}^2 + Q_{vw}^2}}$	Half distance that origin is moved by motor $\mathbf{Q}$.
Flector $\mathbf{F}$	$\widehat{\lVert\mathbf{F}\rVert} = \sqrt{\dfrac{F_{px}^2 + F_{py}^2 + F_{pz}^2 + F_{gw}^2}{F_{pw}^2 + F_{gx}^2 + F_{gy}^2 + F_{gz}^2}}$	Half distance that origin is moved by flector $\mathbf{F}$.

Euclidean Measurements

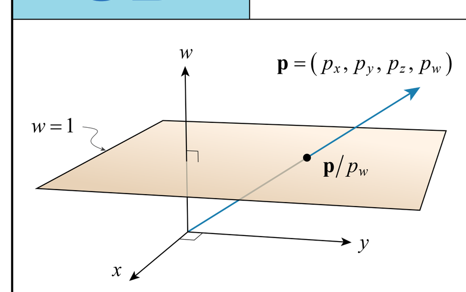
Distance d Between Objects	Cosine of Angle ϕ Between Objects
$d(\mathbf{a},\mathbf{b}) = \lVert\operatorname{att}(\mathbf{a} \wedge \mathbf{b})\rVert_\bullet + \lVert\mathbf{a} \wedge \operatorname{att}(\mathbf{b})\rVert_\circ$	$\cos\phi(\mathbf{a},\mathbf{b}) = \lVert\mathbf{a} \vee \mathbf{b}^\star\rVert_\bullet + \lVert\mathbf{a}\rVert_\circ \lVert\mathbf{b}\rVert_\circ$
$d(\mathbf{a},\mathbf{b}) = \mathbf{a} \vee \mathbf{b} + \lVert\mathbf{a} \wedge \operatorname{att}(\mathbf{b})\rVert_\circ$, if $\operatorname{gr}(\mathbf{a}) + \operatorname{gr}(\mathbf{b}) = n$	$\cos\phi(\mathbf{a},\mathbf{b}) = \mathbf{a} \vee \mathbf{b}^\star + \lVert\mathbf{a}\rVert_\circ \lVert\mathbf{b}\rVert_\circ$, if $\operatorname{gr}(\mathbf{a}) = \operatorname{gr}(\mathbf{b})$

Transformation Groups



Point $\mathbf{p}$ (Vector) — 0D

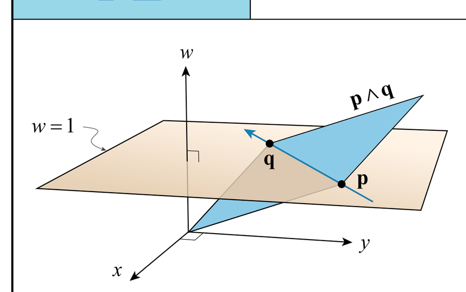
$\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4$ — Position: $p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$; Weight: $p_w\mathbf{e}_4$



Bulk	$\mathbf{p}_\bullet = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$
Weight	$\mathbf{p}_\circ = p_w\mathbf{e}_4$
Bulk dual	$\mathbf{p}^\star = p_x\mathbf{e}_{423} + p_y\mathbf{e}_{431} + p_z\mathbf{e}_{412}$
Weight dual	$\mathbf{p}^{☆} = p_w\mathbf{e}_{321}$
Bulk norm	$\lVert\mathbf{p}\rVert_\bullet = \mathbf{1}\sqrt{p_x^2 + p_y^2 + p_z^2}$
Weight norm	$\lVert\mathbf{p}\rVert_\circ = \lvert p_w \rvert\,\mathbb{1}$
Attitude	$\operatorname{att}(\mathbf{p}) = \mathbf{p} \vee \overline{\mathbf{e}}_4 = p_w\mathbf{1}$
Right complement	$\overline{\mathbf{p}} = p_x\mathbf{e}_{423} + p_y\mathbf{e}_{431} + p_z\mathbf{e}_{412} + p_w\mathbf{e}_{321}$
Degrees of freedom	$\operatorname{DOF}(3,0) = 3$

Line $\boldsymbol{l}$ (Bivector) — 1D

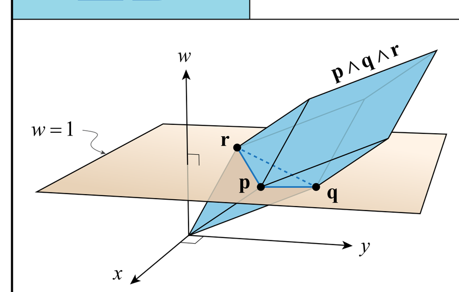
$\boldsymbol{l} = l_{vx}\mathbf{e}_{41} + l_{vy}\mathbf{e}_{42} + l_{vz}\mathbf{e}_{43} + l_{mx}\mathbf{e}_{23} + l_{my}\mathbf{e}_{31} + l_{mz}\mathbf{e}_{12}$ — Direction: $l_{vx}\mathbf{e}_{41} + l_{vy}\mathbf{e}_{42} + l_{vz}\mathbf{e}_{43}$; Moment: $l_{mx}\mathbf{e}_{23} + l_{my}\mathbf{e}_{31} + l_{mz}\mathbf{e}_{12}$



Bulk	$\boldsymbol{l}_\bullet = l_{mx}\mathbf{e}_{23} + l_{my}\mathbf{e}_{31} + l_{mz}\mathbf{e}_{12}$
Weight	$\boldsymbol{l}_\circ = l_{vx}\mathbf{e}_{41} + l_{vy}\mathbf{e}_{42} + l_{vz}\mathbf{e}_{43}$
Bulk dual	$\boldsymbol{l}^\star = -l_{mx}\mathbf{e}_{41} - l_{my}\mathbf{e}_{42} - l_{mz}\mathbf{e}_{43}$
Weight dual	$\boldsymbol{l}^{☆} = -l_{vx}\mathbf{e}_{23} - l_{vy}\mathbf{e}_{31} - l_{vz}\mathbf{e}_{12}$
Bulk norm	$\lVert\boldsymbol{l}\rVert_\bullet = \mathbf{1}\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}$
Weight norm	$\lVert\boldsymbol{l}\rVert_\circ = \mathbb{1}\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}$
Attitude	$\operatorname{att}(\boldsymbol{l}) = \boldsymbol{l} \vee \overline{\mathbf{e}}_4 = l_{vx}\mathbf{e}_1 + l_{vy}\mathbf{e}_2 + l_{vz}\mathbf{e}_3$
Right complement	$\overline{\boldsymbol{l}} = -l_{mx}\mathbf{e}_{41} - l_{my}\mathbf{e}_{42} - l_{mz}\mathbf{e}_{43} - l_{vx}\mathbf{e}_{23} - l_{vy}\mathbf{e}_{31} - l_{vz}\mathbf{e}_{12}$
Degrees of freedom	$\operatorname{DOF}(3,1) = 4$
Constraints	$\boldsymbol{l}_{\mathbf{v}} \cdot \boldsymbol{l}_{\mathbf{m}} = 0$

Plane $\mathbf{g}$ (Trivector) — 2D

$\mathbf{g} = g_x\mathbf{e}_{423} + g_y\mathbf{e}_{431} + g_z\mathbf{e}_{412} + g_w\mathbf{e}_{321}$ — Normal: $g_x\mathbf{e}_{423} + g_y\mathbf{e}_{431} + g_z\mathbf{e}_{412}$; Position: $g_w\mathbf{e}_{321}$



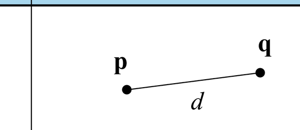
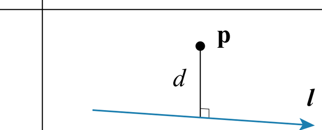
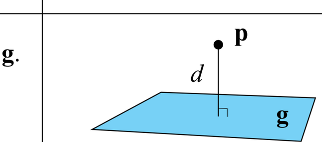
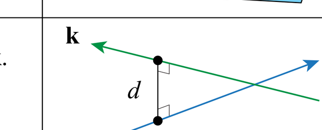
Bulk	$\mathbf{g}_\bullet = g_w\mathbf{e}_{321}$
Weight	$\mathbf{g}_\circ = g_x\mathbf{e}_{423} + g_y\mathbf{e}_{431} + g_z\mathbf{e}_{412}$
Bulk dual	$\mathbf{g}^\star = -g_w\mathbf{e}_4$
Weight dual	$\mathbf{g}^{☆} = -g_x\mathbf{e}_1 - g_y\mathbf{e}_2 - g_z\mathbf{e}_3$
Bulk norm	$\lVert\mathbf{g}\rVert_\bullet = \lvert g_w \rvert\,\mathbf{1}$
Weight norm	$\lVert\mathbf{g}\rVert_\circ = \mathbb{1}\sqrt{g_x^2 + g_y^2 + g_z^2}$
Attitude	$\operatorname{att}(\mathbf{g}) = \mathbf{g} \vee \overline{\mathbf{e}}_4 = g_x\mathbf{e}_{23} + g_y\mathbf{e}_{31} + g_z\mathbf{e}_{12}$
Right complement	$\overline{\mathbf{g}} = -g_x\mathbf{e}_1 - g_y\mathbf{e}_2 - g_z\mathbf{e}_3 - g_w\mathbf{e}_4$
Degrees of freedom	$\operatorname{DOF}(3,2) = 3$

Geometric Product $\mathbf{a} ⟑ \mathbf{b}$

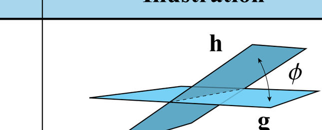
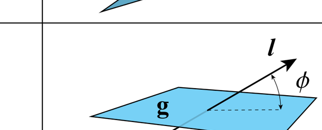
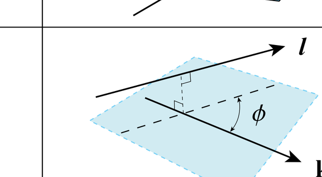
(Highlighted: $\mathbf{a} \wedge \mathbf{b} = \mathbf{a} ⟑ \mathbf{b}$)

a \ b	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_4$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbb{1}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{23}$	$\mathbf{e}_{423}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{412}$	$-\mathbf{e}_4$	$-\mathbf{e}_{423}$	$\mathbf{e}_3$	$-\mathbf{e}_{321}$	$-\mathbf{e}_1$	$-\mathbf{e}_{43}$	$\mathbb{1}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{431}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$-\mathbf{e}_{43}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_4$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{e}_{321}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$\mathbb{1}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{412}$
$\mathbf{e}_4$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	0	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	0	0	0	$\mathbb{1}$	0
$\mathbf{e}_{41}$	$\mathbf{e}_{41}$	$\mathbf{e}_4$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	0	0	0	0	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	0	0	0	$-\mathbf{e}_{423}$	0
$\mathbf{e}_{42}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	$\mathbf{e}_4$	$\mathbf{e}_{423}$	0	0	0	0	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	0	0	0	$-\mathbf{e}_{431}$	0
$\mathbf{e}_{43}$	$\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$\mathbf{e}_4$	0	0	0	0	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	0	0	0	$-\mathbf{e}_{412}$	0
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_{423}$	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	-1	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$-\mathbf{e}_4$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$\mathbf{e}_1$	$\mathbf{e}_{41}$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$	$-\mathbf{e}_{321}$	$-\mathbf{e}_1$	$\mathbf{e}_{431}$	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{12}$	-1	$-\mathbf{e}_{23}$	$\mathbf{e}_{412}$	$-\mathbf{e}_4$	$-\mathbf{e}_{423}$	$\mathbf{e}_2$	$\mathbf{e}_{42}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	-1	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_4$	$\mathbf{e}_3$	$\mathbf{e}_{43}$
$\mathbf{e}_{423}$	$\mathbf{e}_{423}$	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	0	0	0	0	$-\mathbf{e}_4$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	0	0	0	$\mathbf{e}_{41}$	0
$\mathbf{e}_{431}$	$\mathbf{e}_{431}$	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	0	0	0	0	$\mathbf{e}_{412}$	$-\mathbf{e}_4$	$-\mathbf{e}_{423}$	0	0	0	$\mathbf{e}_{42}$	0
$\mathbf{e}_{412}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	0	0	0	0	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_4$	0	0	0	$\mathbf{e}_{43}$	0
$\mathbf{e}_{321}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	-1	$\mathbf{e}_4$
$\mathbb{1}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	0	0	0	0	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	$-\mathbf{e}_4$	0

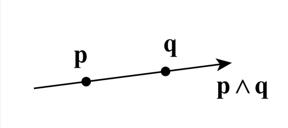
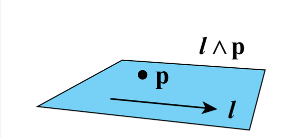
DISTANCE

Distance Formula	Illustration
Distance d between points $\mathbf{p}$ and $\mathbf{q}$. $d(\mathbf{p},\mathbf{q}) = \lVert\mathbf{q}_{xyz}p_w - \mathbf{p}_{xyz}q_w\rVert\,\mathbf{1} + \lvert p_w q_w \rvert\,\mathbb{1}$	
Perpendicular distance d between point $\mathbf{p}$ and line $\boldsymbol{l}$. $d(\mathbf{p},\boldsymbol{l}) = \lVert\boldsymbol{l}_{\mathbf{v}} \times \mathbf{p}_{xyz} + p_w\boldsymbol{l}_{\mathbf{m}}\rVert\,\mathbf{1} + \lVert p_w\boldsymbol{l}_{\mathbf{v}}\rVert\,\mathbb{1}$	
Perpendicular distance d between point $\mathbf{p}$ and plane $\mathbf{g}$. $d(\mathbf{p},\mathbf{g}) = (\mathbf{p} \cdot \mathbf{g})\,\mathbf{1} + \lVert p_w\mathbf{g}_{xyz}\rVert\,\mathbb{1}$	
Perpendicular distance d between skew lines $\boldsymbol{l}$ and $\mathbf{k}$. $d(\boldsymbol{l},\mathbf{k}) = -(\boldsymbol{l}_{\mathbf{v}} \cdot \mathbf{k}_{\mathbf{m}} + \boldsymbol{l}_{\mathbf{m}} \cdot \mathbf{k}_{\mathbf{v}})\,\mathbf{1} + \lVert\boldsymbol{l}_{\mathbf{v}} \times \mathbf{k}_{\mathbf{v}}\rVert\,\mathbb{1}$	

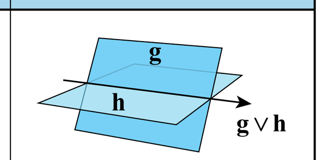
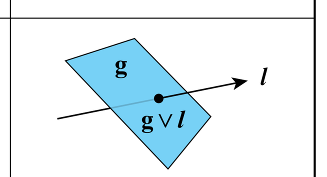
ANGLE

Angle Formula	Illustration
Cosine of angle ϕ between planes $\mathbf{g}$ and $\mathbf{h}$. $\cos\phi(\mathbf{g},\mathbf{h}) = (\mathbf{g}_{xyz} \cdot \mathbf{h}_{xyz})\,\mathbf{1} + \lVert\mathbf{g}\rVert_\circ\lVert\mathbf{h}\rVert_\circ$	
Cosine of angle ϕ between plane $\mathbf{g}$ and line $\boldsymbol{l}$. $\cos\phi(\mathbf{g},\boldsymbol{l}) = \lVert\mathbf{g}_{xyz} \times \boldsymbol{l}_{\mathbf{v}}\rVert\,\mathbf{1} + \lVert\mathbf{g}\rVert_\circ\lVert\boldsymbol{l}\rVert_\circ$	
Cosine of angle ϕ between lines $\boldsymbol{l}$ and $\mathbf{k}$. $\cos\phi(\boldsymbol{l},\mathbf{k}) = (\boldsymbol{l}_{\mathbf{v}} \cdot \mathbf{k}_{\mathbf{v}})\,\mathbf{1} + \lVert\boldsymbol{l}\rVert_\circ\lVert\mathbf{k}\rVert_\circ$	

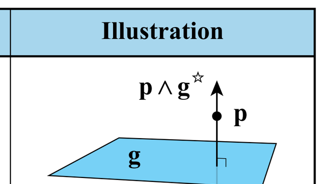
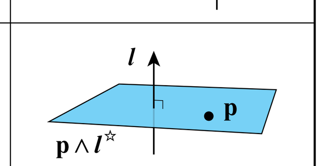
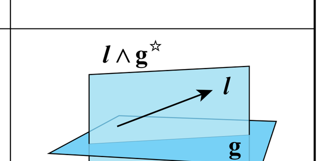
JOIN

Join Operation	Illustration
Line containing points $\mathbf{p}$ and $\mathbf{q}$. $\mathbf{p} \wedge \mathbf{q} = (p_w q_x - p_x q_w)\mathbf{e}_{41} + (p_w q_y - p_y q_w)\mathbf{e}_{42} + (p_w q_z - p_z q_w)\mathbf{e}_{43} + (p_y q_z - p_z q_y)\mathbf{e}_{23} + (p_z q_x - p_x q_z)\mathbf{e}_{31} + (p_x q_y - p_y q_x)\mathbf{e}_{12}$	
Plane containing line $\boldsymbol{l}$ and point $\mathbf{p}$. $\boldsymbol{l} \wedge \mathbf{p} = (l_{vy}p_z - l_{vz}p_y + l_{mx}p_w)\mathbf{e}_{423} + (l_{vz}p_x - l_{vx}p_z + l_{my}p_w)\mathbf{e}_{431} + (l_{vx}p_y - l_{vy}p_x + l_{mz}p_w)\mathbf{e}_{412} - (l_{mx}p_x + l_{my}p_y + l_{mz}p_z)\mathbf{e}_{321}$	

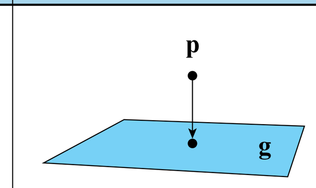
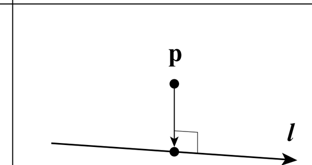
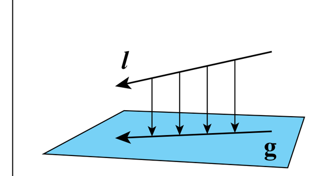
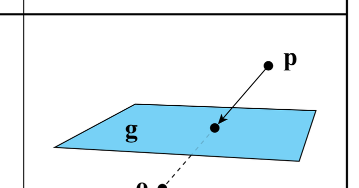
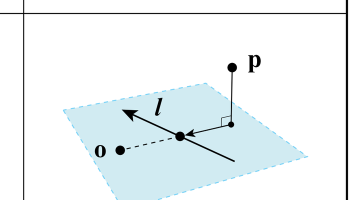
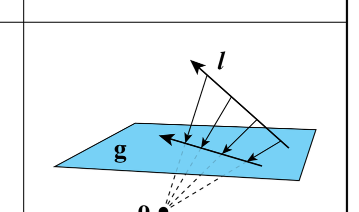
MEET

Meet Operation	Illustration
Line where planes $\mathbf{g}$ and $\mathbf{h}$ intersect. $\mathbf{g} \vee \mathbf{h} = (g_z h_y - g_y h_z)\mathbf{e}_{41} + (g_x h_z - g_z h_x)\mathbf{e}_{42} + (g_y h_x - g_x h_y)\mathbf{e}_{43} + (g_x h_w - g_w h_x)\mathbf{e}_{23} + (g_y h_w - g_w h_y)\mathbf{e}_{31} + (g_z h_w - g_w h_z)\mathbf{e}_{12}$	
Point where plane $\mathbf{g}$ and line $\boldsymbol{l}$ intersect. $\mathbf{g} \vee \boldsymbol{l} = (g_z l_{my} - g_y l_{mz} + g_w l_{vx})\mathbf{e}_1 + (g_x l_{mz} - g_z l_{mx} + g_w l_{vy})\mathbf{e}_2 + (g_y l_{mx} - g_x l_{my} + g_w l_{vz})\mathbf{e}_3 - (g_x l_{vx} + g_y l_{vy} + g_z l_{vz})\mathbf{e}_4$	

EXPANSION

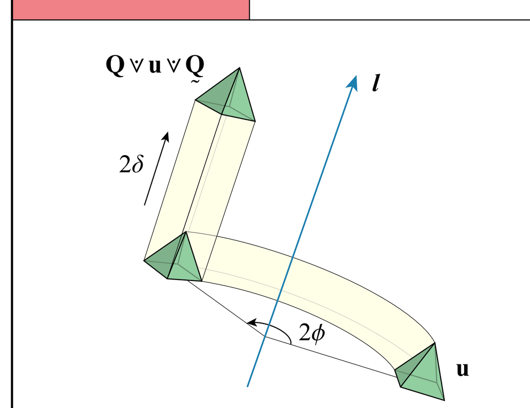
Expansion Operation	Illustration
Line containing point $\mathbf{p}$ and orthogonal to plane $\mathbf{g}$. $\mathbf{p} \wedge \mathbf{g}^{☆} = -p_w g_x\mathbf{e}_{41} - p_w g_y\mathbf{e}_{42} - p_w g_z\mathbf{e}_{43} + (p_z g_y - p_y g_z)\mathbf{e}_{23} + (p_x g_z - p_z g_x)\mathbf{e}_{31} + (p_y g_x - p_x g_y)\mathbf{e}_{12}$	
Plane containing point $\mathbf{p}$ and orthogonal to line $\boldsymbol{l}$. $\mathbf{p} \wedge \boldsymbol{l}^{☆} = -p_w l_{vx}\mathbf{e}_{423} - p_w l_{vy}\mathbf{e}_{431} - p_w l_{vz}\mathbf{e}_{412} + (p_x l_{vx} + p_y l_{vy} + p_z l_{vz})\mathbf{e}_{321}$	
Plane containing line $\boldsymbol{l}$ and orthogonal to plane $\mathbf{g}$. $\boldsymbol{l} \wedge \mathbf{g}^{☆} = (l_{vy}g_z - l_{vz}g_y)\mathbf{e}_{423} + (l_{vz}g_x - l_{vx}g_z)\mathbf{e}_{431} + (l_{vx}g_y - l_{vy}g_x)\mathbf{e}_{412} - (l_{mx}g_x + l_{my}g_y + l_{mz}g_z)\mathbf{e}_{321}$	

PROJECTION

Projection Operation	Illustration
Orthogonal projection of point $\mathbf{p}$ onto plane $\mathbf{g}$. $\mathbf{g} \vee (\mathbf{p} \wedge \mathbf{g}^{☆}) = (g_x^2 + g_y^2 + g_z^2)(p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4) - (g_x p_x + g_y p_y + g_z p_z + g_w p_w)(g_x\mathbf{e}_1 + g_y\mathbf{e}_2 + g_z\mathbf{e}_3)$	
Orthogonal projection of point $\mathbf{p}$ onto line $\boldsymbol{l}$. $\boldsymbol{l} \vee (\mathbf{p} \wedge \boldsymbol{l}^{☆}) = (l_{vx}p_x + l_{vy}p_y + l_{vz}p_z)(l_{vx}\mathbf{e}_1 + l_{vy}\mathbf{e}_2 + l_{vz}\mathbf{e}_3) + (l_{vx}^2 + l_{vy}^2 + l_{vz}^2)\,p_w\mathbf{e}_4 + (l_{vy}l_{mz} - l_{vz}l_{my})\,p_w\mathbf{e}_1 + (l_{vz}l_{mx} - l_{vx}l_{mz})\,p_w\mathbf{e}_2 + (l_{vx}l_{my} - l_{vy}l_{mx})\,p_w\mathbf{e}_3$	
Orthogonal projection of line $\boldsymbol{l}$ onto plane $\mathbf{g}$. $\mathbf{g} \vee (\boldsymbol{l} \wedge \mathbf{g}^{☆}) = (g_x^2 + g_y^2 + g_z^2)(l_{vx}\mathbf{e}_{41} + l_{vy}\mathbf{e}_{42} + l_{vz}\mathbf{e}_{43}) - (g_x l_{vx} + g_y l_{vy} + g_z l_{vz})(g_x\mathbf{e}_{41} + g_y\mathbf{e}_{42} + g_z\mathbf{e}_{43}) + (g_x l_{mx} + g_y l_{my} + g_z l_{mz})(g_x\mathbf{e}_{23} + g_y\mathbf{e}_{31} + g_z\mathbf{e}_{12}) + (g_z l_{vy} - g_y l_{vz})\,g_w\mathbf{e}_{23} + (g_x l_{vz} - g_z l_{vx})\,g_w\mathbf{e}_{31} + (g_y l_{vx} - g_x l_{vy})\,g_w\mathbf{e}_{12}$	
Central projection of point $\mathbf{p}$ onto plane $\mathbf{g}$. $\mathbf{g} \vee (\mathbf{p} \wedge \mathbf{g}^\star) = g_w^2(p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3) - (g_x p_x + g_y p_y + g_z p_z)\,g_w\mathbf{e}_4$	
Central projection of point $\mathbf{p}$ onto line $\boldsymbol{l}$. $\boldsymbol{l} \vee (\mathbf{p} \wedge \boldsymbol{l}^\star) = (l_{mx}^2 + l_{my}^2 + l_{mz}^2)(p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3) - (l_{mx}p_x + l_{my}p_y + l_{mz}p_z)(l_{mx}\mathbf{e}_1 + l_{my}\mathbf{e}_2 + l_{mz}\mathbf{e}_3) + (l_{mx}(l_{vz}p_y - l_{vy}p_z) + l_{my}(l_{vx}p_z - l_{vz}p_x) + l_{mz}(l_{vy}p_x - l_{vx}p_y))\,\mathbf{e}_4$	
Central projection of line $\boldsymbol{l}$ onto plane $\mathbf{g}$. $\mathbf{g} \vee (\boldsymbol{l} \wedge \mathbf{g}^\star) = (g_y l_{mz} - g_z l_{my})\,g_w\mathbf{e}_{41} + g_w^2 l_{mx}\mathbf{e}_{23} + (g_z l_{mx} - g_x l_{mz})\,g_w\mathbf{e}_{42} + g_w^2 l_{my}\mathbf{e}_{31} + (g_x l_{my} - g_y l_{mx})\,g_w\mathbf{e}_{43} + g_w^2 l_{mz}\mathbf{e}_{12}$	

Motor $\mathbf{Q}$ — Motion Operator

$\mathbf{Q} = Q_{vx}\mathbf{e}_{41} + Q_{vy}\mathbf{e}_{42} + Q_{vz}\mathbf{e}_{43} + Q_{vw}\mathbb{1} + Q_{mx}\mathbf{e}_{23} + Q_{my}\mathbf{e}_{31} + Q_{mz}\mathbf{e}_{12} + Q_{mw}\mathbf{1}$ — Rotation: $Q_{vx}\mathbf{e}_{41} + Q_{vy}\mathbf{e}_{42} + Q_{vz}\mathbf{e}_{43} + Q_{vw}\mathbb{1}$; Moment and Displacement: $Q_{mx}\mathbf{e}_{23} + Q_{my}\mathbf{e}_{31} + Q_{mz}\mathbf{e}_{12} + Q_{mw}\mathbf{1}$



$\mathbf{Q} = \exp_{⟇}\left[(\delta\mathbf{1} + \phi\mathbb{1}) ⟇ \boldsymbol{l}\right] = \cos_{⟇}(\delta\mathbf{1} + \phi\mathbb{1}) + \sin_{⟇}(\delta\mathbf{1} + \phi\mathbb{1}) ⟇ \boldsymbol{l}$

$\mathbf{Q} = \boldsymbol{l}\sin\phi - \boldsymbol{l}^{☆}\delta\cos\phi - \delta\sin\phi + \mathbb{1}\cos\phi$

$\mathbf{Q} = (l_{vx}\mathbf{e}_{41} + l_{vy}\mathbf{e}_{42} + l_{vz}\mathbf{e}_{43} + l_{mx}\mathbf{e}_{23} + l_{my}\mathbf{e}_{31} + l_{mz}\mathbf{e}_{12})\sin\phi + (l_{vx}\mathbf{e}_{23} + l_{vy}\mathbf{e}_{31} + l_{vz}\mathbf{e}_{12})\,\delta\cos\phi - \delta\sin\phi + \mathbb{1}\cos\phi$

$\mathbf{Q} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{Q}}$ rotates object $\mathbf{u}$ through angle 2ϕ about line $\boldsymbol{l}$ and translates by distance 2δ along direction of line $\boldsymbol{l}$.

$$\sqrt{\mathbf{Q}} = \frac{\mathbf{Q} + \mathbb{1}}{\sqrt{2 + 2Q_{\mathbb{1}}}} ⟇ \left(\mathbb{1} - \frac{Q_{\mathbf{1}}}{2 + 2Q_{\mathbb{1}}}\right)$$

Bulk	$\mathbf{Q}_\bullet = Q_{mx}\mathbf{e}_{23} + Q_{my}\mathbf{e}_{31} + Q_{mz}\mathbf{e}_{12} + Q_{mw}\mathbf{1}$
Weight	$\mathbf{Q}_\circ = Q_{vx}\mathbf{e}_{41} + Q_{vy}\mathbf{e}_{42} + Q_{vz}\mathbf{e}_{43} + Q_{vw}\mathbb{1}$
Bulk dual	$\mathbf{Q}^\star = -Q_{mx}\mathbf{e}_{41} - Q_{my}\mathbf{e}_{42} - Q_{mz}\mathbf{e}_{43} + Q_{mw}\mathbb{1}$
Weight dual	$\mathbf{Q}^{☆} = -Q_{vx}\mathbf{e}_{23} - Q_{vy}\mathbf{e}_{31} - Q_{vz}\mathbf{e}_{12} + Q_{vw}\mathbf{1}$
Bulk norm	$\lVert\mathbf{Q}\rVert_\bullet = \mathbf{1}\sqrt{Q_{mx}^2 + Q_{my}^2 + Q_{mz}^2 + Q_{mw}^2}$
Weight norm	$\lVert\mathbf{Q}\rVert_\circ = \mathbb{1}\sqrt{Q_{vx}^2 + Q_{vy}^2 + Q_{vz}^2 + Q_{vw}^2}$

$\mathbf{R} = (a_x\mathbf{e}_{41} + a_y\mathbf{e}_{42} + a_z\mathbf{e}_{43})\sin\phi + \mathbb{1}\cos\phi$	$\mathbf{R} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{R}}$ rotates object $\mathbf{u}$ through angle 2ϕ about unit axis $\mathbf{a}$ through origin.
$\mathbf{S} = \boldsymbol{l}\sin\phi + \mathbb{1}\cos\phi$	$\mathbf{S} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{S}}$ rotates object $\mathbf{u}$ through angle 2ϕ about unitized line $\boldsymbol{l}$.
$\mathbf{T} = \tau_x\mathbf{e}_{23} + \tau_y\mathbf{e}_{31} + \tau_z\mathbf{e}_{12} + \mathbb{1}$	$\mathbf{T} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{T}}$ translates object $\mathbf{u}$ by vector $2\boldsymbol{\tau}$.
Line $\boldsymbol{l}$	$\boldsymbol{l} ⟇ \mathbf{u} ⟇ \underset{\sim}{\boldsymbol{l}}$ rotates object $\mathbf{u}$ through 180° about unitized line $\boldsymbol{l}$.

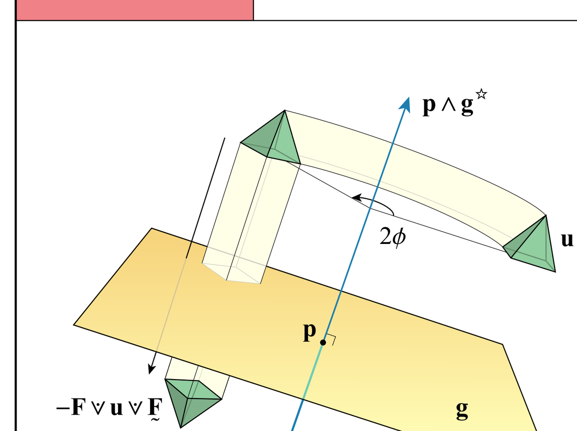
Matrix conversion: $\mathbf{M}_{\mathbf{Q}} = \mathbf{A}_{\mathbf{Q}} + \mathbf{B}_{\mathbf{Q}}$ $\qquad \mathbf{M}_{\mathbf{Q}}^{-1} = \mathbf{A}_{\mathbf{Q}} - \mathbf{B}_{\mathbf{Q}}$

$$\mathbf{A}_{\mathbf{Q}} = \begin{bmatrix} 1 - 2(Q_{vy}^2 + Q_{vz}^2) & 2Q_{vx}Q_{vy} & 2Q_{vz}Q_{vx} & 2(Q_{vy}Q_{mz} - Q_{vz}Q_{my}) \\ 2Q_{vx}Q_{vy} & 1 - 2(Q_{vz}^2 + Q_{vx}^2) & 2Q_{vy}Q_{vz} & 2(Q_{vz}Q_{mx} - Q_{vx}Q_{mz}) \\ 2Q_{vz}Q_{vx} & 2Q_{vy}Q_{vz} & 1 - 2(Q_{vx}^2 + Q_{vy}^2) & 2(Q_{vx}Q_{my} - Q_{vy}Q_{mx}) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B}_{\mathbf{Q}} = \begin{bmatrix} 0 & -2Q_{vz}Q_{vw} & 2Q_{vy}Q_{vw} & 2(Q_{vw}Q_{mx} - Q_{vx}Q_{mw}) \\ 2Q_{vz}Q_{vw} & 0 & -2Q_{vx}Q_{vw} & 2(Q_{vw}Q_{my} - Q_{vy}Q_{mw}) \\ -2Q_{vy}Q_{vw} & 2Q_{vx}Q_{vw} & 0 & 2(Q_{vw}Q_{mz} - Q_{vz}Q_{mw}) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Flector $\mathbf{F}$ — Reflection Operator

$\mathbf{F} = F_{px}\mathbf{e}_1 + F_{py}\mathbf{e}_2 + F_{pz}\mathbf{e}_3 + F_{pw}\mathbf{e}_4 + F_{gx}\mathbf{e}_{423} + F_{gy}\mathbf{e}_{431} + F_{gz}\mathbf{e}_{412} + F_{gw}\mathbf{e}_{321}$ — Point: $F_{px}\mathbf{e}_1 + F_{py}\mathbf{e}_2 + F_{pz}\mathbf{e}_3 + F_{pw}\mathbf{e}_4$; Plane: $F_{gx}\mathbf{e}_{423} + F_{gy}\mathbf{e}_{431} + F_{gz}\mathbf{e}_{412} + F_{gw}\mathbf{e}_{321}$



$\mathbf{F} = \mathbf{p}\sin\phi + \mathbf{g}\cos\phi$

$\mathbf{F} = (p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4)\sin\phi + (g_x\mathbf{e}_{423} + g_y\mathbf{e}_{431} + g_z\mathbf{e}_{412} + g_w\mathbf{e}_{321})\cos\phi$

$-\mathbf{F} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{F}}$ rotates object $\mathbf{u}$ through angle 2ϕ about line perpendicular to plane $\mathbf{g}$ passing through point $\mathbf{p}$ and reflects across plane $\mathbf{g}$.

Bulk	$\mathbf{F}_\bullet = F_{px}\mathbf{e}_1 + F_{py}\mathbf{e}_2 + F_{pz}\mathbf{e}_3 + F_{gw}\mathbf{e}_{321}$
Weight	$\mathbf{F}_\circ = F_{pw}\mathbf{e}_4 + F_{gx}\mathbf{e}_{423} + F_{gy}\mathbf{e}_{431} + F_{gz}\mathbf{e}_{412}$
Bulk dual	$\mathbf{F}^\star = -F_{gw}\mathbf{e}_4 + F_{px}\mathbf{e}_{423} + F_{py}\mathbf{e}_{431} + F_{pz}\mathbf{e}_{412}$
Weight dual	$\mathbf{F}^{☆} = -F_{gx}\mathbf{e}_1 - F_{gy}\mathbf{e}_2 - F_{gz}\mathbf{e}_3 + F_{pw}\mathbf{e}_{321}$
Bulk norm	$\lVert\mathbf{F}\rVert_\bullet = \mathbf{1}\sqrt{F_{px}^2 + F_{py}^2 + F_{pz}^2 + F_{gw}^2}$
Weight norm	$\lVert\mathbf{F}\rVert_\circ = \mathbb{1}\sqrt{F_{pw}^2 + F_{gx}^2 + F_{gy}^2 + F_{gz}^2}$

$\mathbf{H} = v_x\mathbf{e}_1 + v_y\mathbf{e}_2 + v_z\mathbf{e}_3 + \mathbf{g}$	$-\mathbf{H} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{H}}$ translates object $\mathbf{u}$ by vector $2\mathbf{g}_{xyz} \times \mathbf{v}$ and reflects across unitized plane $\mathbf{g}$ (transflection).
Point $\mathbf{p}$	$-\mathbf{p} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{p}}$ inverts object $\mathbf{u}$ through point $\mathbf{p}$.
Plane $\mathbf{g}$	$-\mathbf{g} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{g}}$ reflects object $\mathbf{u}$ across plane $\mathbf{g}$.

Matrix conversion: $\mathbf{M}_{\mathbf{F}} = \mathbf{A}_{\mathbf{F}} + \mathbf{B}_{\mathbf{F}}$ $\qquad \mathbf{M}_{\mathbf{F}}^{-1} = \mathbf{A}_{\mathbf{F}} - \mathbf{B}_{\mathbf{F}}$

$$\mathbf{A}_{\mathbf{F}} = \begin{bmatrix} 2(F_{gy}^2 + F_{gz}^2) - 1 & -2F_{gx}F_{gy} & -2F_{gz}F_{gx} & 2(F_{px}F_{pw} - F_{gx}F_{gw}) \\ -2F_{gx}F_{gy} & 2(F_{gz}^2 + F_{gx}^2) - 1 & -2F_{gy}F_{gz} & 2(F_{py}F_{pw} - F_{gy}F_{gw}) \\ -2F_{gz}F_{gx} & -2F_{gy}F_{gz} & 2(F_{gx}^2 + F_{gy}^2) - 1 & 2(F_{pz}F_{pw} - F_{gz}F_{gw}) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B}_{\mathbf{F}} = \begin{bmatrix} 0 & 2F_{gz}F_{pw} & -2F_{gy}F_{pw} & 2(F_{gy}F_{pz} - F_{gz}F_{py}) \\ -2F_{gz}F_{pw} & 0 & 2F_{gx}F_{pw} & 2(F_{gz}F_{px} - F_{gx}F_{pz}) \\ 2F_{gy}F_{pw} & -2F_{gx}F_{pw} & 0 & 2(F_{gx}F_{py} - F_{gy}F_{px}) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$