# Space-Antispace Transform Correspondence in Projective Geometric Algebra

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The concept of duality can be understood geometrically in an $n$-dimensional projective setting by considering both the subspace that an object occupies and the complementary subspace that the object concurrently does not occupy. The dimensionalities of these two components always sum to $n$, and they represent the *space* and *antispace* associated with the object.[1] The example shown in Figure 1 demonstrates the duality between homogeneous points and lines in a three-dimensional projective space. The triplet of coordinates $(p_x, p_y, p_z)$ can be interpreted as a vector pointing from the origin toward a specific location on the projection plane $z = 1$. This vector corresponds to the one-dimensional space of the point that it represents. The dual of a point materializes when we consider all of the directions of space that are orthogonal to the single direction $(p_x, p_y, p_z)$. As illustrated by the figure, these directions span an $(n-1)$-dimensional subspace that intersects the projection plane at a line when $n = 3$. In this way, the coordinates $(p_x, p_y, p_z)$ can be interpreted as both a point and a line, and they are *duals* of each other.

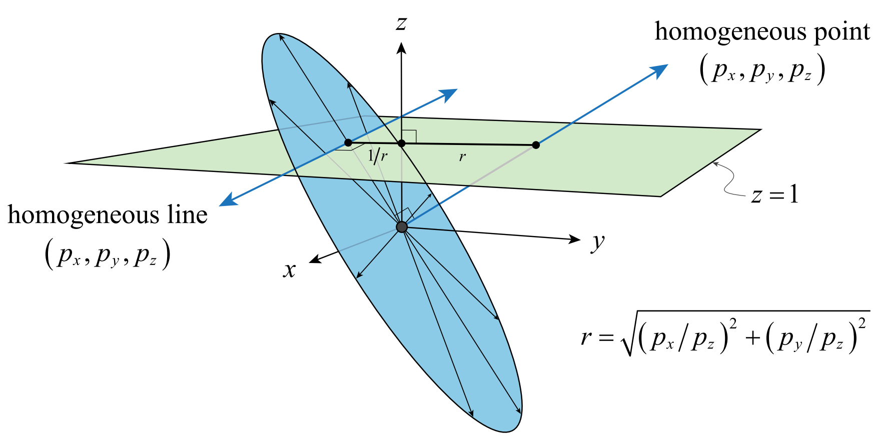


**Figure 1.** The coordinates $(p_x, p_y, p_z)$ can be interpreted as the one-dimensional span of a single vector representing a homogeneous point or as the $(n-1)$-dimensional span of all orthogonal vectors representing a homogeneous hyperplane, which is a line when $n = 3$. Geometrically, these two interpretations are dual to each other, and their distances to the origin are reciprocals of each other.

When we express the coordinates $(p_x, p_y, p_z)$ on the vector basis as $p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$, it explicitly states that we are working with a single spatial dimension representing a point, and the ambiguity is removed. Similarly, if we express the coordinates on the bivector basis as $p_x\mathbf{e}_{23} + p_y\mathbf{e}_{31} + p_z\mathbf{e}_{12}$, then we are working with the two orthogonal spatial dimensions representing a line. In each case, the subscripts of the basis elements tell us which basis vectors are present in the representation, and this defines the *space* of the object. The subscripts also tell us which basis vectors are absent in the representation, and this defines the *antispace* of the object. Acknowledging the existence of both the space and the antispace of any object and assigning equal meaningfulness to them allows us to explore the nature of duality to its fullest. A vector $p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$ is never only a point, but both a point and a line simultaneously, where the point exists in space, and the line exists in antispace. Likewise, a bivector $p_x\mathbf{e}_{23} + p_y\mathbf{e}_{31} + p_z\mathbf{e}_{12}$ is never only a

[1] Antispace is also known as negative space or counterspace.

line, but both a line and a point simultaneously, where the line exists in space, and the point exists in antispace. If we study only the spatial facet of these objects and their higher-dimensional counterparts, then we are missing half of a bigger picture.

It is particularly interesting to consider the Euclidean isometries that map $n$-dimensional space onto itself while preserving distances and angles. We know how each isometry transforms the space of a point, line, plane, etc., but for a complete understanding of the geometry, we must ask ourselves what happens to the antispace of those objects at the same time. Equivalently, when an object is transformed by an isometry, we would like to know how its dual is transformed. The answer requires that we first look at the invariants associated with each Euclidean isometry.

In the two-dimensional plane, the Euclidean isometries consist of a rotation about point, a translation in a specific direction, and a transflection with respect to a specific mirroring line. A reflection is a special case of transflection in which there is no motion parallel to the line, and translation is a special case of rotation in which the center lies in the horizon. Naturally, the invariant of a rotation is its center point, and the invariant of a transflection is its mirroring line. These objects are mapped onto themselves by their associated transforms, and this necessitates that their duals also be mapped onto themselves by whatever corresponding transforms occur in antispace. Because a rotation fixes its center point, the corresponding transform in antispace must fix the line that is dual to that point. And because a transflection fixes its mirroring line, the corresponding transform in antispace must fix the point that is dual to that line. These transforms occurring in antispace are projective analogs of the transforms occurring in space. There is a direct correspondence between the two transforms, and they are inextricably linked. Whenever one transform is applied in space, the other is applied in antispace, and vice-versa.

Figure 2 shows a two-dimensional rotation transform and its projective analog. The green point represents the center of rotation, and the green line is the dual of that point. Under the regular rotation, the center point is fixed, and under the projective rotation, the line dual to the center point is fixed. These are not, however, the only fixed geometries. A regular rotation also fixes the horizon line, and thus its dual, the origin, must be fixed in the projective rotation. This is illustrated by the red point in the figure, which is the focus of the various conic-section orbits. Here, the green line is the directrix.

Figure 3 shows a two-dimensional reflection transform and its projective analog. The green line represents the mirroring plane of the reflection, and the green point is the dual of that line. Under the regular reflection, the mirroring line is fixed, and under the projective reflection, the point dual to the mirroring line is fixed. As with rotation, there are additional fixed geometries under these transforms. A regular reflection fixes the point in the horizon in the direction perpendicular to the mirroring plane. The dual of this point is a line parallel to the mirroring line and containing the origin that remains fixed by the projective reflection. This is illustrated by the red line in the figure, which is clearly a reflection boundary in a sense.

Finally, Figure 4 shows a two-dimensional translation transform and its projective analog. As mentioned above, a translation is a special case of rotation in which the center lies in the horizon. As such, there is no finite fixed geometry that can be shown in the figure for a regular translation. However, the dual of the center in the horizon must be a line containing the origin that is fixed by the projective translation, and that is illustrated by the red line in the figure. Projective translation is especially important because it is the one to which we can most easily assign some practical meaning. It is a *perspective projection* onto the line through the origin perpendicular to the direction of translation.

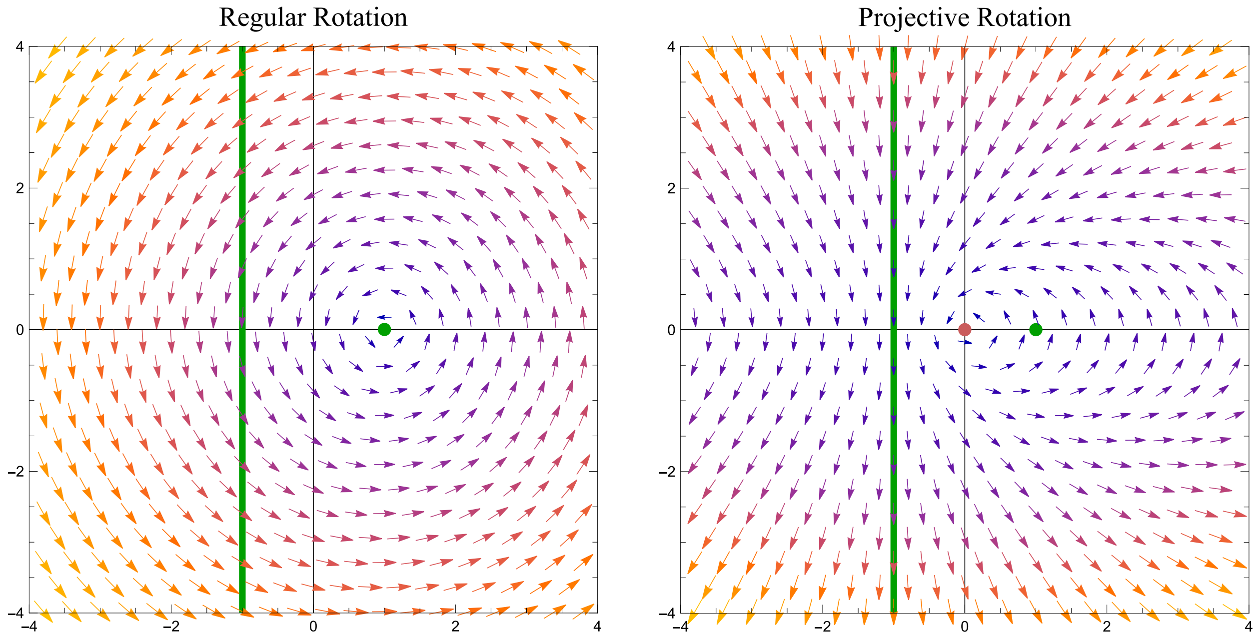


**Figure 2.** (Left) A regular rotation fixes the green center point at $(1,0)$ and the horizon. (Right) The corresponding projective rotation fixes the green line at $x=-1$ dual to the center point and the origin.

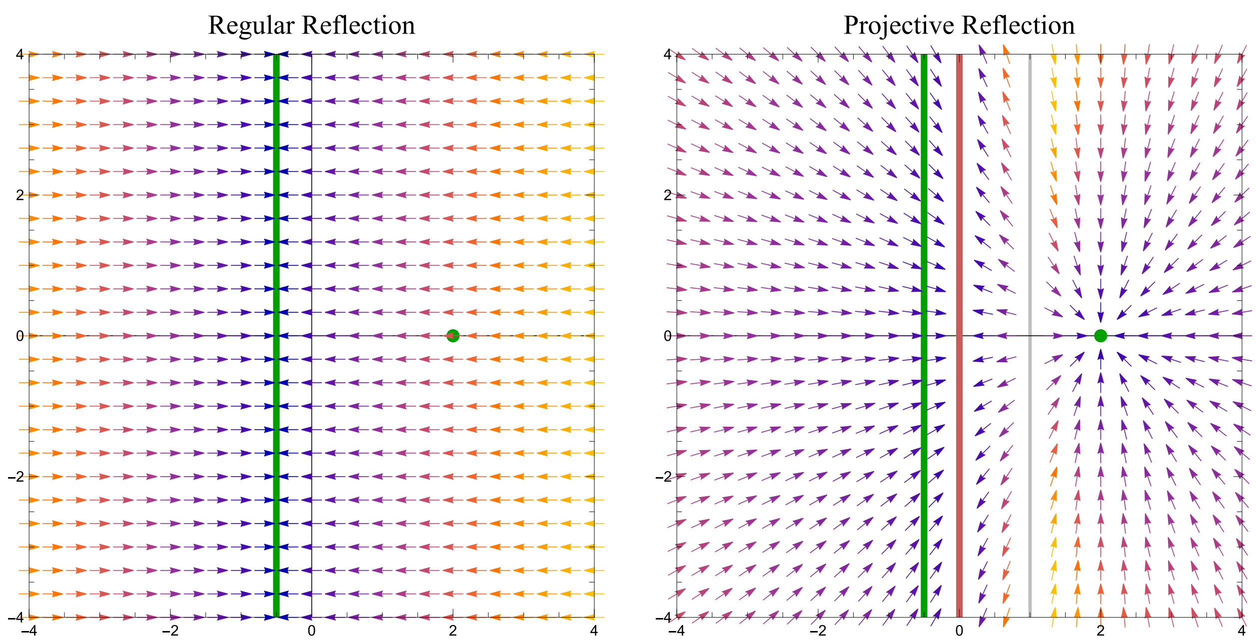


**Figure 3.** (Left) A regular reflection fixes the green mirroring line at $x=-\frac{1}{2}$ and the point in the horizon in the perpendicular direction. (Right) The corresponding projective reflection fixes the green point $(2,0)$ dual to the mirroring line and the line through the origin parallel to the mirroring line.

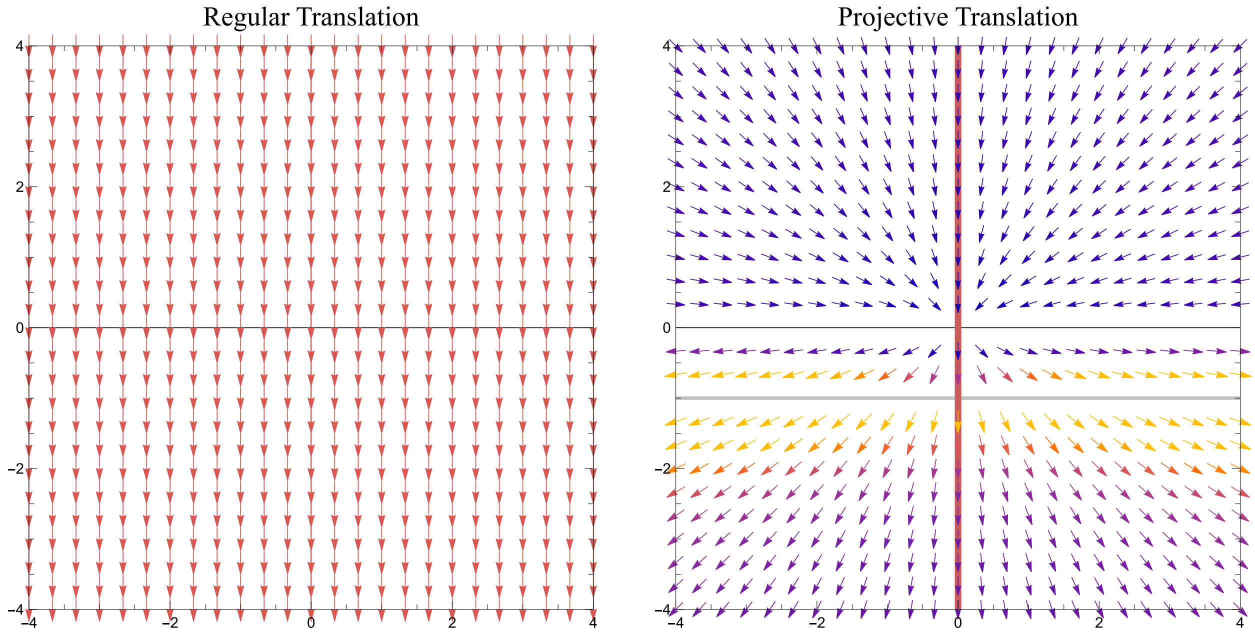


**Figure 4.** (Left) A regular translation fixes the point in the horizon perpendicular to the direction of translation and every line parallel to the direction of translation. (Right) The corresponding projective translation (a perspective projection) fixes the line parallel to the direction of translation through the origin and every point in the line through the origin perpendicular to the direction of translation.

In the three-dimensional projective geometric algebra $\mathcal{G}(2,0,1)$, a homogeneous representation of two-dimensional space, a regular rotation about a center point $\mathbf{c}$ is given by

$$\mathbf{Q} = c_x \mathbf{e}_1 + c_y \mathbf{e}_2 + c_z \mathbf{e}_3 + r\mathbb{1}, \tag{1}$$

and this becomes a translation when $c_z = 0$. A regular transflection across the line $\mathbf{h}$ is given by

$$\mathbf{G} = s + h_x \mathbf{e}_{23} + h_y \mathbf{e}_{31} + h_z \mathbf{e}_{12}, \tag{2}$$

and this becomes a pure reflection when $s = 0$. Together, these operators include all possible Euclidean isometries in the two-dimensional plane. Under the geometric antiproduct $\veebar$, arbitrary products of these operators form the group $E(2)$ with $\mathbb{1}$ as the identity, and they covariantly transform any object $a$ in the algebra through the sandwich products

$$a' = \mathbf{Q} \veebar a \veebar \underset{\sim}{\mathbf{Q}} \quad \text{and} \quad a' = \mathbf{G} \veebar a \veebar \underset{\sim}{\mathbf{G}}. \tag{3}$$

Symmetrically, a projective rotation about the line $\mathbf{c}$ is given by

$$\mathbb{Q} = c_x \mathbf{e}_{23} + c_y \mathbf{e}_{31} + c_z \mathbf{e}_{12} - r, \tag{4}$$

and a projective transflection across the point $\mathbf{h}$ is given by

$$\mathbb{G} = h_x \mathbf{e}_1 + h_y \mathbf{e}_2 + h_z \mathbf{e}_3 - s\mathbb{1}. \tag{5}$$

These two operators generate a different group of transformations under the geometric product $\wedge\!\!\!\!\wedge$. Arbitrary products of these operators form the projective Euclidean group $PE(2)$ with $\mathbf{1}$ as the identity, and they covariantly transform any object $a$ in the algebra through the sandwich products

$$a' = \mathbb{Q} \wedge\!\!\!\!\wedge\, a \wedge\!\!\!\!\wedge\, \tilde{\mathbb{Q}} \quad \text{and} \quad a' = \mathbb{G} \wedge\!\!\!\!\wedge\, a \wedge\!\!\!\!\wedge\, \tilde{\mathbb{G}}. \tag{6}$$

The groups $E(2)$ and $PE(2)$ are isomorphic, and they each contain the orthogonal group $O(2)$ as a common subgroup. The complement operation provides a two-way mapping between transforms associated with members of $E(2)$ and $PE(2)$.

The invariant geometries of the four types of transforms described above are summarized in Table 1. The Euclidean isometries always fix a coinvariant contained in the horizon, and the corresponding projective transforms always fix a coinvariant containing the origin. In general, if $x$ is the primary invariant of a Euclidean isometry, then the complement of the weight of $x$ gives the coinvariant. Symmetrically, if $x$ is the primary invariant of a projective transform, then the complement of the bulk of $x$ gives the coinvariant. When the primary invariant of a Euclidean isometry contains the origin, there is a corresponding projective transform that performs the same operation. Symmetrically, when the primary invariant of a projective transform is contained in the horizon, there is a corresponding Euclidean isometry that performs the same operation. These are where $E(2)$ and $PE(2)$ intersect at $O(2)$.

| **Transform** | **Primary Invariant** | **Coinvariant** |
|---|---|---|
| Regular rotation<br>$\mathbf{Q} = c_x\mathbf{e}_1 + c_y\mathbf{e}_2 + c_z\mathbf{e}_3 + r\mathbb{1}$ | Point $c_x\mathbf{e}_1 + c_y\mathbf{e}_2 + c_z\mathbf{e}_3$ | Horizon line $\mathbf{e}_{12}$ |
| Regular transflection<br>$\mathbf{G} = s + h_x\mathbf{e}_{23} + h_y\mathbf{e}_{31} + h_z\mathbf{e}_{12}$ | Line $h_x\mathbf{e}_{23} + h_y\mathbf{e}_{31} + h_z\mathbf{e}_{12}$ | Point in horizon $h_x\mathbf{e}_1 + h_y\mathbf{e}_2$ |
| Projective rotation<br>$\mathbb{Q} = c_x\mathbf{e}_{23} + c_y\mathbf{e}_{31} + c_z\mathbf{e}_{12} - r$ | Line $c_x\mathbf{e}_{23} + c_y\mathbf{e}_{31} + c_z\mathbf{e}_{12}$ | Origin point $\mathbf{e}_3$ |
| Projective transflection<br>$\mathbb{G} = h_x\mathbf{e}_1 + h_y\mathbf{e}_2 + h_z\mathbf{e}_3 - s\mathbb{1}$ | Point $h_x\mathbf{e}_1 + h_y\mathbf{e}_2 + h_z\mathbf{e}_3$ | Line through origin $h_x\mathbf{e}_{23} + h_y\mathbf{e}_{31}$ |

**Table 1.** These are the invariants of transforms occurring in the 3D projective geometric algebra representing the 2D plane. The primary invariant of any regular transform (a Euclidean isometry) or projective transform is given by the vector or bivector components of the operator itself. The coinvariant is given by the weight complement of the primary invariant in the case of regular transforms and by the bulk complement of the primary invariant in the case of projective transforms.

In the four-dimensional projective geometric algebra $\mathcal{G}(3,0,1)$ representing three-dimensional space, every Euclidean isometry is either a screw transform $\mathbf{Q}$ or a rotoreflection $\mathbf{G}$.[2] The primary invariant of a screw transform is its bivector components, which corresponds to the line about which a rotation is taking place. A rotoreflection can have two primary invariants, one associated with its vector components and a second associated with its trivector components. These invariants, the invariants of the corresponding projective transforms, and the coinvariants for each are summarized in Table 2.

[2] The only geometrical difference between these is that the displacement along the rotation axis in a screw transform is replaced by a reflection in a plane perpendicular to the rotation axis in a rotoreflection.

| Transform | Primary Invariants | Coinvariants |
|---|---|---|
| Regular screw transform <br> $\mathbf{Q} = r_x \mathbf{e}_{41} + r_y \mathbf{e}_{42} + r_z \mathbf{e}_{43} + r_w 𝟙$ <br> $+ u_x \mathbf{e}_{23} + u_y \mathbf{e}_{31} + u_z \mathbf{e}_{12} + u_w$ | Line <br> $r_x \mathbf{e}_{41} + r_y \mathbf{e}_{42} + r_z \mathbf{e}_{43} + u_x \mathbf{e}_{23} + u_y \mathbf{e}_{31} + u_z \mathbf{e}_{12}$ | Line in horizon <br> $r_x \mathbf{e}_{23} + r_y \mathbf{e}_{31} + r_z \mathbf{e}_{12}$ |
| Regular rotoreflection <br> $\mathbf{G} = s_x \mathbf{e}_1 + s_y \mathbf{e}_2 + s_z \mathbf{e}_3 + s_w \mathbf{e}_4$ <br> $+ h_x \mathbf{e}_{234} + h_y \mathbf{e}_{314} + h_z \mathbf{e}_{124} + h_w \mathbf{e}_{321}$ | Plane $h_x \mathbf{e}_{234} + h_y \mathbf{e}_{314} + h_z \mathbf{e}_{124} + h_w \mathbf{e}_{321}$, <br> Point $s_x \mathbf{e}_1 + s_y \mathbf{e}_2 + s_z \mathbf{e}_3 + s_w \mathbf{e}_4$ | Point in horizon <br> $h_x \mathbf{e}_1 + h_y \mathbf{e}_2 + h_z \mathbf{e}_3$, <br> Horizon $\mathbf{e}_{321}$ |
| Projective screw transform <br> $\boldsymbol{\mathfrak{Q}} = u_x \mathbf{e}_{41} + u_y \mathbf{e}_{42} + u_z \mathbf{e}_{43} - u_w 𝟙$ <br> $+ r_x \mathbf{e}_{23} + r_y \mathbf{e}_{31} + r_z \mathbf{e}_{12} - r_w$ | Line <br> $u_x \mathbf{e}_{41} + u_y \mathbf{e}_{42} + u_z \mathbf{e}_{43} + r_x \mathbf{e}_{23} + r_y \mathbf{e}_{31} + r_z \mathbf{e}_{12}$ | Line through origin <br> $r_x \mathbf{e}_{41} + r_y \mathbf{e}_{42} + r_z \mathbf{e}_{43}$ |
| Projective rotoreflection <br> $\boldsymbol{\mathfrak{G}} = h_x \mathbf{e}_1 + h_y \mathbf{e}_2 + h_z \mathbf{e}_3 + h_w \mathbf{e}_4$ <br> $+ s_x \mathbf{e}_{234} + s_y \mathbf{e}_{314} + s_z \mathbf{e}_{124} + s_w \mathbf{e}_{321}$ | Point $h_x \mathbf{e}_1 + h_y \mathbf{e}_2 + h_z \mathbf{e}_3 + h_w \mathbf{e}_4$, <br> Plane $s_x \mathbf{e}_{234} + s_y \mathbf{e}_{314} + s_z \mathbf{e}_{124} + s_w \mathbf{e}_{321}$ | Plane through origin <br> $h_x \mathbf{e}_{234} + h_y \mathbf{e}_{314} + h_z \mathbf{e}_{124}$, <br> Origin $\mathbf{e}_4$ |

**Table 2.** These are the invariants of transforms occurring in the 4D projective geometric algebra representing 3D space. The primary invariant of any regular transform (a Euclidean isometry) or projective transform is given by the vector, bivector, or trivector components of the operator itself. The coinvariants are given by the weight complement of each primary invariant in the case of regular transforms and by the bulk complement of each primary invariant in the case of projective transforms.

The unit-weight invertible elements of the $(n+1)$-dimensional projective geometric algebra $\mathcal{G}(n,0,1)$ constitute a double cover of both the groups $E(n)$ and $PE(n)$. The geometric product corresponds to transform composition in the group $PE(n)$, and the geometric antiproduct corresponds to transform composition in the group $E(n)$. Regular reflections across planes are represented by antivectors (having antigrade one), and they meet at lower-dimensional invariants under the geometric antiproduct. Symmetrically, projective reflections across points are represented by vectors (having grade one), and they join at higher-dimensional invariants under the geometric product. A sandwich product

$$\mathbf{Q} ⟑ a ⟑ \tilde{\mathbf{Q}} \tag{7}$$

transforms the space of $a$ with an element of $PE(n)$, and it transforms the antispace of $a$ with the complementary element of $E(n)$. Symmetrically, a sandwich product

$$\mathbf{Q} ⟇ a ⟇ \underset{\sim}{\mathbf{Q}} \tag{8}$$

transforms the space of $a$ with an element of $E(n)$, and it transforms the antispace of $a$ with the complementary element of $PE(n)$.

The groups $E(n)$ and $PE(n)$ have a number of subgroups, and the hierarchical relationships among them are shown in Figure 5. In particular, the Euclidean group $E(n)$ contains the special Euclidean subgroup $SE(n)$ consisting of all combinations of regular rotations, which are covered by the antigrade-2 elements of $\mathcal{G}(n,0,1)$. Correspondingly, the projective Euclidean group $PE(n)$ contains the projective special Euclidean subgroup $PSE(n)$ consisting of all combinations of projective rotations, which are covered by the grade-2 elements of $\mathcal{G}(n,0,1)$. The subgroups $SE(n)$ and $PSE(n)$ further contain translation subgroups $T(n)$ and $PT(n)$, respectively.

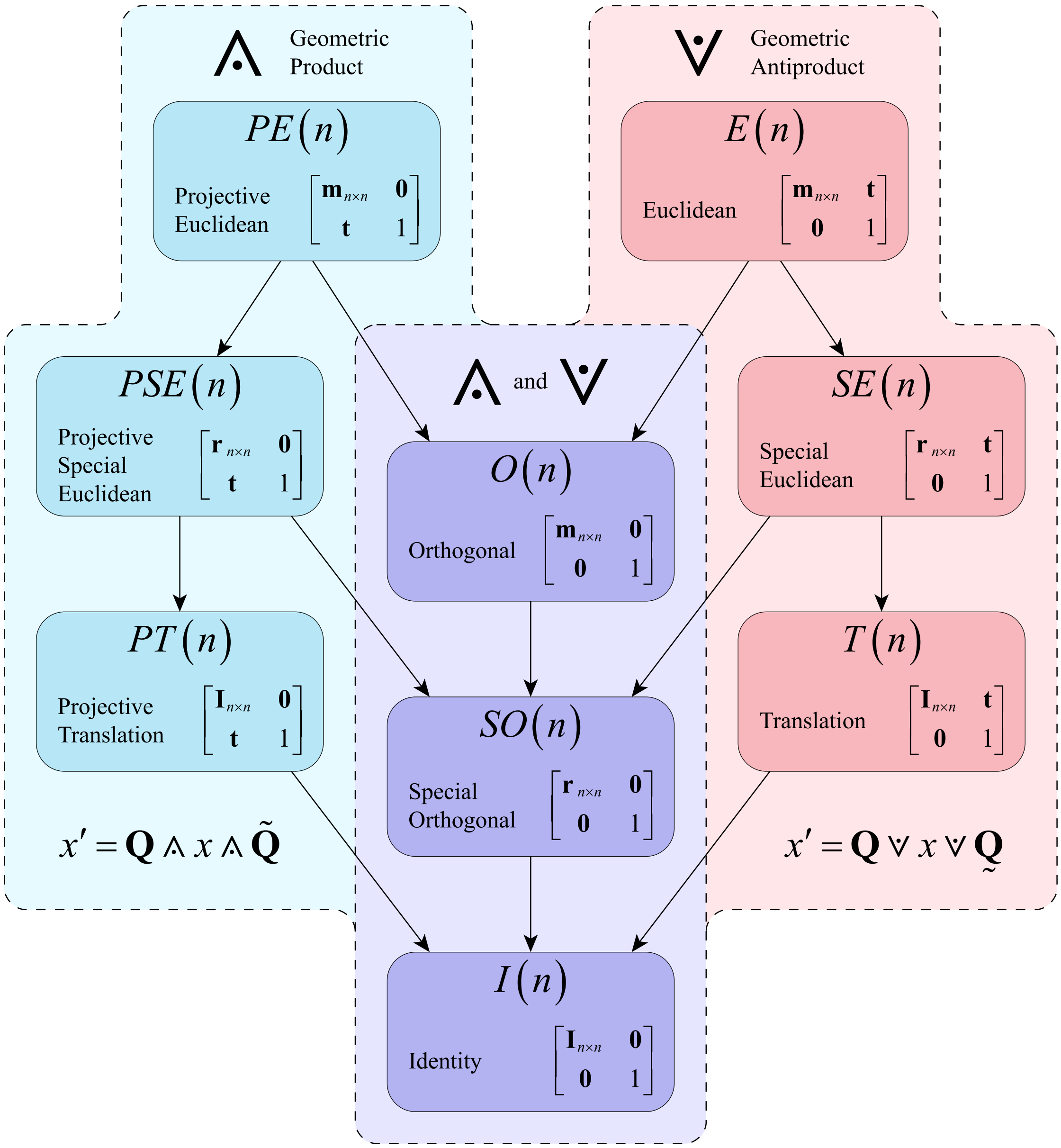


**Figure 5.** The unit-weight invertible elements of the projective geometric algebra $\mathcal{G}(n,0,1)$ are a double cover of both the Euclidean group $E(n)$ and the projective Euclidean group $PE(n)$. Elements corresponding to transforms in $E(n)$ are composed with the geometric antiproduct, and elements corresponding to transforms in $PE(n)$ are composed with the geometric product. Transforms belonging to the common subgroup $O(n)$ have two representations in $\mathcal{G}(n,0,1)$, one associated with the geometric product, and one associated with the geometric antiproduct.

Transforms about invariants containing the origin are the same in both $E(n)$ and $PE(n)$, and they constitute the common subgroup $O(n)$. Every member of $O(n)$ has a representation in $\mathcal{G}(n,0,1)$ that transforms elements with the geometric product and a complementary representation that transforms elements with the geometric antiproduct. For example, in $\mathcal{G}(3,0,1)$, a conventional quaternion $\mathbf{q}$ can be expressed as

$$\mathbf{q} = x\mathbf{e}_{23} + y\mathbf{e}_{31} + z\mathbf{e}_{12} - w\mathbf{1}, \tag{9}$$

which covariantly transforms any object $a$ with the sandwich product $\mathbf{q} \wedge a \wedge \tilde{\mathbf{q}}$, and it can be expressed as

$$\mathbf{q} = x\mathbf{e}_{41} + y\mathbf{e}_{42} + z\mathbf{e}_{43} + w\mathbb{1}, \tag{10}$$

which covariantly transforms any object $a$ with the sandwich product $\mathbf{q} \vee a \vee \mathbf{q}$.

In terms of matrix multiplication, a general element of the group $E(n)$ transforms a point by multiplying on the left by an $(n+1)\times(n+1)$ matrix of the form

$$\begin{bmatrix} \mathbf{m}_{n\times n} & \mathbf{t}_{n\times 1} \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix}, \tag{11}$$

where the $n \times n$ submatrix $\mathbf{m}$ is orthogonal. A general element of the corresponding group $PE(n)$ transforms points with matrices of the form

$$\begin{bmatrix} \mathbf{m}_{n\times n} & \mathbf{0}_{n\times 1} \\ \mathbf{t}_{1\times n} & 1 \end{bmatrix}. \tag{12}$$

In the special subgroups of $E(n)$ and $PE(n)$, the submatrix $\mathbf{m}$ has a determinant of $+1$. In the translation subgroups $T(n)$ and $PT(n)$, $\mathbf{m}$ is the identity matrix. Finally, when $\mathbf{t} = \mathbf{0}$, the matrices in Equations (11) and (12) have the same form and belong to $O(n)$.

The isomorphic mapping between $E(n)$ and $PE(n)$ is given by the inverse transpose operation on the matrix representatives. That is, if $\mathbf{M}$ is an $(n+1)\times(n+1)$ matrix representing an element of $E(n)$, then the corresponding element of $PE(n)$ is given by $(\mathbf{M}^{-1})^{\mathrm{T}}$. Of course, this operation is an involution, and the mapping works both ways.